OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) **4757**

Further Applications of Advanced Mathematics (FP3)

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- Graph paper
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:
None

Monday 1 June 2009
Morning

Duration: 1 hour 30 minutes

4757

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer any **three** questions.
- Do **not** write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

Option 1: Vectors

1 The point A $(-1, 12, 5)$ lies on the plane P with equation $8x - 3y + 10z = 6$. The point B $(6, -2, 9)$ lies on the plane Q with equation $3x - 4y - 2z = 8$. The planes P and Q intersect in the line L.

(i) Find an equation for the line L. **[5]**

(ii) Find the shortest distance between L and the line AB. **[6]**

The lines M and N are both parallel to L, with M passing through A and N passing through B.

(iii) Find the distance between the parallel lines M and N. **[5]**

The point C has coordinates $(k, 0, 2)$, and the line AC intersects the line N at the point D.

(iv) Find the value of k, and the coordinates of D. **[8]**

Option 2: Multi-variable calculus

2 A surface has equation $z = 3x(x + y)^3 - 2x^3 + 24x$.

(i) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. **[4]**

(ii) Find the coordinates of the three stationary points on the surface. **[7]**

(iii) Find the equation of the normal line at the point P $(1, -2, 19)$ on the surface. **[3]**

(iv) The point Q $(1 + k, -2 + h, 19 + 3h)$ is on the surface and is close to P. Find an approximate expression for k in terms of h. **[4]**

(v) Show that there is only one point on the surface at which the tangent plane has an equation of the form $27x - z = d$. Find the coordinates of this point and the corresponding value of d. **[6]**

Option 3: Differential geometry

3 A curve has parametric equations $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, for $0 \leqslant \theta \leqslant \pi$, where a is a positive constant.

(i) Show that the arc length s from the origin to a general point on the curve is given by $s = 4a\sin\frac{1}{2}\theta$. **[6]**

(ii) Find the intrinsic equation of the curve giving s in terms of a and ψ, where $\tan\psi = \dfrac{\mathrm{d}y}{\mathrm{d}x}$. **[4]**

(iii) Hence, or otherwise, show that the radius of curvature at a point on the curve is $4a\cos\frac{1}{2}\theta$. **[3]**

(iv) Find the coordinates of the centre of curvature corresponding to the point on the curve where $\theta = \frac{2}{3}\pi$. **[6]**

(v) Find the area of the surface generated when the curve is rotated through 2π radians about the x-axis. **[5]**

Option 4: Groups

4 The group $G = \{1, 2, 3, 4, 5, 6\}$ has multiplication modulo 7 as its operation. The group $H = \{1, 5, 7, 11, 13, 17\}$ has multiplication modulo 18 as its operation.

(i) Show that the groups G and H are both cyclic. **[4]**

(ii) List all the proper subgroups of G. **[3]**

(iii) Specify an isomorphism between G and H. **[4]**

The group $S = \{\mathrm{a, b, c, d, e, f}\}$ consists of functions with domain $\{1, 2, 3\}$ given by

$$\begin{array}{lll} \mathrm{a}(1) = 2 & \mathrm{a}(2) = 3 & \mathrm{a}(3) = 1 \\ \mathrm{b}(1) = 3 & \mathrm{b}(2) = 1 & \mathrm{b}(3) = 2 \\ \mathrm{c}(1) = 1 & \mathrm{c}(2) = 3 & \mathrm{c}(3) = 2 \\ \mathrm{d}(1) = 3 & \mathrm{d}(2) = 2 & \mathrm{d}(3) = 1 \\ \mathrm{e}(1) = 1 & \mathrm{e}(2) = 2 & \mathrm{e}(3) = 3 \\ \mathrm{f}(1) = 2 & \mathrm{f}(2) = 1 & \mathrm{f}(3) = 3 \end{array}$$

and the group operation is composition of functions.

(iv) Show that ad = c and find da. **[4]**

(v) Give a reason why S is not isomorphic to G. **[1]**

(vi) Find the order of each element of S. **[4]**

(vii) List all the proper subgroups of S. **[4]**

[Question 5 is printed overleaf.]

Option 5: Markov chains

This question requires the use of a calculator with the ability to handle matrices.

5 Each level of a fantasy computer game is set in a single location, Alphaworld, Betaworld, Chiworld or Deltaworld. After completing a level, a player goes on to the next level, which could be set in the same location as the previous level, or in a different location.

In the first version of the game, the initial and transition probabilities are as follows.

Level 1 is set in Alphaworld or Betaworld, with probabilities 0.6, 0.4 respectively.

After a level set in Alphaworld, the next level will be set in Betaworld, Chiworld or Deltaworld, with probabilities 0.7, 0.1, 0.2 respectively.

After a level set in Betaworld, the next level will be set in Alphaworld, Betaworld or Deltaworld, with probabilities 0.1, 0.8, 0.1 respectively.

After a level set in Chiworld, the next level will also be set in Chiworld.

After a level set in Deltaworld, the next level will be set in Alphaworld, Betaworld or Chiworld, with probabilities 0.3, 0.6, 0.1 respectively.

The situation is modelled as a Markov chain with four states.

(i) Write down the transition matrix. **[2]**

(ii) Find the probabilities that level 14 is set in each location. **[3]**

(iii) Find the probability that level 15 is set in the same location as level 14. **[3]**

(iv) Find the level at which the probability of being set in Chiworld first exceeds 0.5. **[3]**

(v) Following a level set in Betaworld, find the expected number of further levels which will be set in Betaworld before changing to a different location. **[3]**

In the second version of the game, the initial probabilities and the transition probabilities after Alphaworld, Betaworld and Deltaworld are all the same as in the first version; but after a level set in Chiworld, the next level will be set in Chiworld or Deltaworld, with probabilities 0.9, 0.1 respectively.

(vi) By considering powers of the new transition matrix, or otherwise, find the equilibrium probabilities for the four locations. **[5]**

In the third version of the game, the initial probabilities and the transition probabilities after Alphaworld, Betaworld and Deltaworld are again all the same as in the first version; but the transition probabilities after Chiworld have changed again. The equilibrium probabilities for Alphaworld, Betaworld, Chiworld and Deltaworld are now 0.11, 0.75, 0.04, 0.1 respectively.

(vii) Find the new transition probabilities after a level set in Chiworld. **[5]**

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1 (i)	Putting $x=0$, $-3y+10z=6$, $-4y-2z=8$ $y=-2$, $z=0$ Direction is given by $\begin{pmatrix}8\\-3\\10\end{pmatrix}\times\begin{pmatrix}3\\-4\\-2\end{pmatrix}$ $=\begin{pmatrix}46\\46\\-23\end{pmatrix}$ Equation of *L* is $\mathbf{r}=\begin{pmatrix}0\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\-1\end{pmatrix}$	M1 A1 M1 A1 A1 ft **5**	Finding coords of a point on the line or $(2,\ 0,\ -1)$, $(1,\ -1,\ -\frac{1}{2})$ etc or finding a second point *Dependent on M1M1* Accept any form Condone omission of '**r** ='
(ii)	$\overrightarrow{\mathrm{AB}}\times\mathbf{d}=\begin{pmatrix}7\\-14\\4\end{pmatrix}\times\begin{pmatrix}2\\2\\-1\end{pmatrix}=\begin{pmatrix}6\\15\\42\end{pmatrix}$ $\left[\ =3\begin{pmatrix}2\\5\\14\end{pmatrix}\ \right]$ Distance is $\left[\begin{pmatrix}-1\\12\\5\end{pmatrix}-\begin{pmatrix}0\\-2\\0\end{pmatrix}\right].\hat{\mathbf{n}}=\dfrac{\begin{pmatrix}-1\\14\\5\end{pmatrix}.\begin{pmatrix}2\\5\\14\end{pmatrix}}{\sqrt{2^2+5^2+14^2}}$ $=\dfrac{138}{15}=\dfrac{46}{5}=9.2$	M1 A2 ft M1 A1 ft A1 **6**	Evaluating $\overrightarrow{\mathrm{AB}}\times\mathbf{d}$ Give A1 ft if just one error Appropriate scalar product Fully correct expression
(iii)	$\left\vert\overrightarrow{\mathrm{AB}}\times\mathbf{d}\right\vert=\left\vert\begin{pmatrix}6\\15\\42\end{pmatrix}\right\vert=\sqrt{6^2+15^2+42^2}$ Distance is $\dfrac{\left\vert\overrightarrow{\mathrm{AB}}\times\mathbf{d}\right\vert}{\vert\mathbf{d}\vert}=\dfrac{\sqrt{6^2+15^2+42^2}}{\sqrt{2^2+2^2+1^2}}$ $=\dfrac{45}{3}=15$	M1 M1 M1A1 ft A1 **5**	For $\left\vert\overrightarrow{\mathrm{AB}}\times\mathbf{d}\right\vert$ Evaluating magnitude *In this part, M marks are dependent on previous M marks*

(iv)	At D, $\begin{pmatrix}-1\\12\\5\end{pmatrix}+\lambda\begin{pmatrix}k+1\\-12\\-3\end{pmatrix}=\begin{pmatrix}6\\-2\\9\end{pmatrix}+\mu\begin{pmatrix}2\\2\\-1\end{pmatrix}$ $12-12\lambda=-2+2\mu$ $5-3\lambda=9-\mu$ $\lambda=\frac{1}{3},\quad \mu=5$ $-1+\frac{1}{3}(k+1)=6+10$ $k=50$ D is $(6+2\mu,\ -2+2\mu,\ 9-\mu)$ i.e. $(16,\ 8,\ 4)$	M1 A1 ft M1M1 M1 A1 M1 A1 **8**	*Condone use of same parameter on both sides* Two equations for λ and μ Obtaining λ and μ (numerically) Give M1 for λ and μ in terms of *k* Equation for *k* Obtaining coordinates of D

Alternative solutions for Q1

1 (i)	e.g. $23x-23y=46$ $x=t,\ y=t-2$ $3t-4(t-2)-2z=8$ $x=t,\ y=t-2,\ z=-\frac{1}{2}t$	M1A1 M1A1 ft A1 **5**	Eliminating one of *x, y, z*
(ii)	$\overrightarrow{PQ}=\begin{pmatrix}-1+7\mu\\12-14\mu\\5+4\mu\end{pmatrix}-\begin{pmatrix}2\lambda\\-2+2\lambda\\-\lambda\end{pmatrix}\quad \overrightarrow{PQ}\,.\,\mathbf{d}=\overrightarrow{PQ}\,.\,\overrightarrow{AB}=0$ $2(-1+7\mu-2\lambda)+2(14-14\mu-2\lambda)-(5+4\mu+\lambda)=0$ $7(-1+7\mu-2\lambda)-14(14-14\mu-2\lambda)+4(5+4\mu+\lambda)=0\ \ \lambda=27/25,\ \ \mu=47/75$ $\lvert\overrightarrow{PQ}\rvert=\sqrt{(92/75)^2+(230/75)^2+(644/75)^2}=9.2$	M1 A1 ft A1 ft M1A1 ft A1 **6**	Two equations for λ and μ Expression for shortest distance
(iii)	$\overrightarrow{AX}\cdot\mathbf{d}=\begin{pmatrix}6+2\lambda+1\\-2+2\lambda-12\\9-\lambda-5\end{pmatrix}\cdot\begin{pmatrix}2\\2\\-1\end{pmatrix}=0$ $2(7+2\lambda)+2(2\lambda-14)-(4-\lambda)=0$ $\lambda=2$ $\overrightarrow{AX}=\begin{pmatrix}11\\-10\\2\end{pmatrix}$ $AX=\sqrt{11^2+10^2+2^2}$ $=15$	M1 A1 ft M1 M1 A1 **5**	

(iv)	$\begin{pmatrix}7\\-14\\4\end{pmatrix}\cdot\left[\begin{pmatrix}k+1\\-12\\-3\end{pmatrix}\times\begin{pmatrix}2\\2\\-1\end{pmatrix}\right]=0$ $\begin{pmatrix}7\\-14\\4\end{pmatrix}\cdot\begin{pmatrix}18\\k-5\\2k+26\end{pmatrix}=0$ $126-14k+70+8k+104=0$ $k=50$ At D, $\begin{pmatrix}-1\\12\\5\end{pmatrix}+\lambda\begin{pmatrix}51\\-12\\-3\end{pmatrix}=\begin{pmatrix}6\\-2\\9\end{pmatrix}+\mu\begin{pmatrix}2\\2\\-1\end{pmatrix}$ $-1+51\lambda=6+2\mu$ $12-12\lambda=-2+2\mu$ $5-3\lambda=9-\mu$ $\lambda=\frac{1}{3},\quad \mu=5$ D is $(6+2\mu,\ -2+2\mu,\ 9-\mu)$ i.e. $(16,\ 8,\ 4)$	M1 M1 A1 M1 A1 ft M1 M1 A1 **8**	Appropriate scalar triple product equated to zero Equation for k *Condone use of same parameter on both sides* Two equations for λ and μ Obtaining λ or μ Obtaining coordinates of D

2 (i)	$\frac{\partial z}{\partial x} = 3(x+y)^3 + 9x(x+y)^2 - 6x^2 + 24$ $\frac{\partial z}{\partial y} = 9x(x+y)^2$	M1 A2 A1 **4**	Partial differentiation Give A1 if just one minor error
(ii)	At stationary points, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ $9x(x+y)^2 = 0 \Rightarrow x = 0$ or $y = -x$ If $x = 0$ then $3y^3 + 24 = 0$ $y = -2$; one stationary point is $(0,\ -2,\ 0)$ If $y = -x$ then $-6x^2 + 24 = 0$ $x = \pm 2$; stationary points are $(2,\ -2,\ 32)$ and $(-2,\ 2,\ -32)$	M1 M1 A1A1 M1 A1 A1 **7**	If A0A0, give A1 for $x = \pm 2$
(iii)	At P $(1,\ -2,\ 19)$, $\frac{\partial z}{\partial x} = 24$, $\frac{\partial z}{\partial y} = 9$ Normal line is $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 19 \end{pmatrix} + \lambda \begin{pmatrix} 24 \\ 9 \\ -1 \end{pmatrix}$	B1 M1 A1 ft **3**	For normal vector (allow sign error) Condone omission of '**r** ='
(iv)	$\delta z \approx \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$ $= 24\,\delta x + 9\,\delta y$ $3h \approx 24k + 9h$ $k \approx -\frac{1}{4}h$	M1 A1 ft M1 A1 **4**	
	OR Tangent plane is $24x + 9y - z = -13$ $24(1+k) + 9(-2+h) - (19+3h) \approx -13$ M2A1 ft $k \approx -\frac{1}{4}h$ A1		
(v)	$\frac{\partial z}{\partial x} = 27$ and $\frac{\partial z}{\partial y} = 0$ $9x(x+y)^2 = 0 \Rightarrow x = 0$ or $y = -x$ If $x = 0$ then $3y^3 + 24 = 27$ $y = 1,\ z = 0$; point is $(0,\ 1,\ 0)$ $d = 0$ If $y = -x$ then $-6x^2 + 24 = 27$ $x^2 = -\frac{1}{2}$; there are no other points	M1 M1 A1 A1 M1 A1 **6**	(Allow M1 for $\frac{\partial z}{\partial x} = -27$)

3 (i)	$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2=[a(1+\cos\theta)]^2+(a\sin\theta)^2$ $=a^2(2+2\cos\theta)$ $=4a^2\cos^2\frac{1}{2}\theta$ $s=\int 2a\cos\frac{1}{2}\theta\,\mathrm{d}\theta$ $=4a\sin\frac{1}{2}\theta+C$ $s=0$ when $\theta=0 \Rightarrow C=0$	M1 A1 M1 M1 A1 A1 ag **6**	Forming $\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2$ Using half-angle formula Integrating to obtain $k\sin\frac{1}{2}\theta$ Correctly obtained (+C not needed) *Dependent on all previous marks*
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{a\sin\theta}{a(1+\cos\theta)}$ $=\frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2a\cos^2\frac{1}{2}\theta}=\tan\frac{1}{2}\theta$ $\psi=\frac{1}{2}\theta$, and so $s=4a\sin\psi$	M1 M1 A1 A1 **4**	 Using half-angle formulae
(iii)	$\rho=\frac{\mathrm{d}s}{\mathrm{d}\psi}=4a\cos\psi$ $=4a\cos\frac{1}{2}\theta$	M1 A1 ft A1 ag **3**	Differentiating intrinsic equation
	OR $\rho=\frac{\left(4a^2\cos^2\frac{1}{2}\theta\right)^{3/2}}{a(1+\cos\theta)(a\cos\theta)-(-a\sin\theta)(a\sin\theta)}$ M1A1 ft $=\frac{8a^3\cos^3\frac{1}{2}\theta}{a^2(1+\cos\theta)}=\frac{8a^3\cos^3\frac{1}{2}\theta}{2a^2\cos^2\frac{1}{2}\theta}=4a\cos\frac{1}{2}\theta$ A1 ag		Correct expression for ρ or κ
(iv)	When $\theta=\frac{2}{3}\pi,\ \psi=\frac{1}{3}\pi,\ x=a(\frac{2}{3}\pi+\frac{1}{2}\sqrt{3}),\ y=\frac{3}{2}a$ $\rho=2a$ $\hat{\mathbf{n}}=\begin{pmatrix}-\sin\psi\\ \cos\psi\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\end{pmatrix}$ $\mathbf{c}=\begin{pmatrix}a(\frac{2}{3}\pi+\frac{1}{2}\sqrt{3})\\ \frac{3}{2}a\end{pmatrix}+2a\begin{pmatrix}-\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\end{pmatrix}$ Centre of curvature is $(\ a(\frac{2}{3}\pi-\frac{1}{2}\sqrt{3}),\ \frac{5}{2}a\ )$	B1 M1 A1 M1 A1A1 **6**	 Obtaining a normal vector Correct unit normal *(possibly in terms of θ)* Accept $(1.23a,\ 2.5a)$

(v)	Curved surface area is $\int 2\pi\, y\, \mathrm{d}s$ $= \int_0^{\pi} 2\pi\, a(1-\cos\theta)\, 2a\cos\tfrac{1}{2}\theta\, \mathrm{d}\theta$ $= \int_0^{\pi} 8\pi a^2 \sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\, \mathrm{d}\theta$ $= \left[\tfrac{16}{3}\pi a^2 \sin^3\tfrac{1}{2}\theta\right]_0^{\pi}$ $= \tfrac{16}{3}\pi a^2$	M1 A1 ft M1 M1 A1	**5**	Correct integral expression in any form *(including limits; may be implied by later working)* Obtaining an integrable form Obtaining $k\sin^3\tfrac{1}{2}\theta$ *or equivalent*

<table>
<tr><td>4 (i)</td><td>In G, $3^2=2,\ 3^3=6,\ 3^4=4,\ 3^5=5,\ 3^6=1$
[or $5^2=4,\ 5^3=6,\ 5^4=2,\ 5^5=3,\ 5^6=1$]
In H, $5^2=7,\ 5^3=17,\ 5^4=13,\ 5^5=11,\ 5^6=1$
[or $11^2=13,\ 11^3=17,\ 11^4=7,\ 11^5=5,\ 11^6=1$]
G has an element 3 (or 5) of order 6
H has an element 5 (or 11) of order 6</td><td>M1
A1
B1
B1
4</td><td>All powers of an element of order 6
All powers correct in both groups</td></tr>
<tr><td>(ii)</td><td>$\{1,\ 6\}$
$\{1,\ 2,\ 4\}$</td><td>B1
B2
3</td><td>Ignore $\{1\}$ and G
Deduct 1 mark (from B1B2) for each proper subgroup in excess of two</td></tr>
<tr><td>(iii)</td><td>$G \quad H$
$1 \leftrightarrow 1$
$2 \leftrightarrow 7$
$3 \leftrightarrow 5$
$4 \leftrightarrow 13$
$5 \leftrightarrow 11$
$6 \leftrightarrow 17$
OR
$G \quad H$
$1 \leftrightarrow 1$
$2 \leftrightarrow 13$
$3 \leftrightarrow 11$
$4 \leftrightarrow 7$
$5 \leftrightarrow 5$
$6 \leftrightarrow 17$</td><td>B4
4</td><td>Give B3 for 4 correct, B2 for 3 correct,
B1 for 2 correct</td></tr>
<tr><td>(iv)</td><td>$\mathrm{ad}(1)=\mathrm{a}(3)=1$
$\mathrm{ad}(2)=\mathrm{a}(2)=3$
$\mathrm{ad}(3)=\mathrm{a}(1)=2$, so $\mathrm{ad}=\mathrm{c}$
$\mathrm{da}(1)=\mathrm{d}(2)=2$
$\mathrm{da}(2)=\mathrm{d}(3)=1$
$\mathrm{da}(3)=\mathrm{d}(1)=3$, so $\mathrm{da}=\mathrm{f}$</td><td>M1
A1
M1
A1
4</td><td>Evaluating e.g. $\mathrm{ad}(1)$ (one case sufficient; intermediate value must be shown)
For $\mathrm{ad}=\mathrm{c}$ correctly shown
Evaluating e.g. $\mathrm{da}(1)$ (one case sufficient; no need for any working)</td></tr>
<tr><td>(v)</td><td>S is not abelian; G is abelian</td><td>B1
1</td><td>or S has 3 elements of order 2;
G has 1 element of order 2
or S is not cyclic etc</td></tr>
<tr><td>(vi)</td><td><table><tr><td>Element</td><td>a</td><td>b</td><td>c</td><td>d</td><td>e</td><td>f</td></tr><tr><td>Order</td><td>3</td><td>3</td><td>2</td><td>2</td><td>1</td><td>2</td></tr></table></td><td>B4
4</td><td>Give B3 for 5 correct, B2 for 3 correct,
B1 for 1 correct</td></tr>
<tr><td>(vii)</td><td>$\{\mathrm{e},\ \mathrm{c}\},\ \{\mathrm{e},\ \mathrm{d}\},\ \{\mathrm{e},\ \mathrm{f}\}$
$\{\mathrm{e},\ \mathrm{a},\ \mathrm{b}\}$</td><td>B1B1B1
B1
4</td><td>Ignore { e } and S
If more than 4 proper subgroups are given, deduct 1 mark for each proper subgroup in excess of 4</td></tr>
</table>

Pre-multiplication by transition matrix

5 (i)	$\mathbf{P}=\begin{pmatrix}0&0.1&0&0.3\\0.7&0.8&0&0.6\\0.1&0&1&0.1\\0.2&0.1&0&0\end{pmatrix}$	B2 **2**	Give B1 for two columns correct
(ii)	$\mathbf{P}^{13}\begin{pmatrix}0.6\\0.4\\0\\0\end{pmatrix}=\begin{pmatrix}0.0810\\0.5684\\0.2760\\0.0746\end{pmatrix}$	M1 A2 **3**	Using $\mathbf{P}^{13}$ (or $\mathbf{P}^{14}$) Give A1 for 2 probabilities correct *(Max A1 if not at least 3dp)* *Tolerance* ± 0.0001
(iii)	$0.5684\times 0.8+0.2760$ $=0.731$	M1M1 A1 ft **3**	For 0.5684×0.8 and 0.2760 *Accept 0.73 to 0.7312*
(iv)	$\mathbf{P}^{30}\begin{pmatrix}0.6\\0.4\\0\\0\end{pmatrix}=\begin{pmatrix}.\\.\\0.4996\\.\end{pmatrix}$, $\mathbf{P}^{31}\begin{pmatrix}0.6\\0.4\\0\\0\end{pmatrix}=\begin{pmatrix}.\\.\\0.5103\\.\end{pmatrix}$ Level 32	M1 A1 A1 **3**	Finding P(C) for some powers of **P** For identifying $\mathbf{P}^{31}$
(v)	Expected number of levels including the next change of location is $\frac{1}{0.2}=5$ Expected number of further levels in B is 4	M1 A1 A1 **3**	For $1/(1-0.8)$ or $0.8/(1-0.8)$ For 5 or 4 For 4 as final answer
(vi)	$\mathbf{Q}=\begin{pmatrix}0&0.1&0&0.3\\0.7&0.8&0&0.6\\0.1&0&0.9&0.1\\0.2&0.1&0.1&0\end{pmatrix}$ $\mathbf{Q}^n\to\begin{pmatrix}0.0916&0.0916&0.0916&0.0916\\0.6183&0.6183&0.6183&0.6183\\0.1908&0.1908&0.1908&0.1908\\0.0992&0.0992&0.0992&0.0992\end{pmatrix}$ A: 0.0916 B: 0.6183 C: 0.1908 D: 0.0992	B1 M1 M1 A2 **5**	*Can be implied* Evaluating powers of **Q** *or Obtaining (at least) 3 equations from* $\mathbf{Qp}=\mathbf{p}$ Limiting matrix with equal columns *or Solving to obtain one equilib prob* *or M2 for other complete method* Give A1 for two correct *(Max A1 if not at least 3dp)* *Tolerance* ± 0.0001

(vii)	$\begin{pmatrix} 0 & 0.1 & a & 0.3 \\ 0.7 & 0.8 & b & 0.6 \\ 0.1 & 0 & c & 0.1 \\ 0.2 & 0.1 & d & 0 \end{pmatrix}\begin{pmatrix} 0.11 \\ 0.75 \\ 0.04 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0.11 \\ 0.75 \\ 0.04 \\ 0.1 \end{pmatrix}$	M1 A1	Transition matrix and $\begin{pmatrix} 0.11 \\ 0.75 \\ 0.04 \\ 0.1 \end{pmatrix}$
	$0.075 + 0.04a + 0.03 = 0.11$ $0.077 + 0.6 + 0.04b + 0.06 = 0.75$ $0.011 + 0.04c + 0.01 = 0.04$ $0.022 + 0.075 + 0.04d = 0.1$	M1	Forming at least one equation *or* $a+b+c+d=1$
	$a = 0.125,\quad b = 0.325,\quad c = 0.475,\quad d = 0.075$	A2 **5**	Give A1 for two correct

Post-multiplication by transition matrix

5 (i)	$\mathbf{P}=\begin{pmatrix}0 & 0.7 & 0.1 & 0.2\\ 0.1 & 0.8 & 0 & 0.1\\ 0 & 0 & 1 & 0\\ 0.3 & 0.6 & 0.1 & 0\end{pmatrix}$	B2	**2**	Give B1 for two rows correct
(ii)	$\begin{pmatrix}0.6 & 0.4 & 0 & 0\end{pmatrix}\mathbf{P}^{13}$ $=\begin{pmatrix}0.0810 & 0.5684 & 0.2760 & 0.0746\end{pmatrix}$	M1 A2	**3**	Using $\mathbf{P}^{13}$ (or $\mathbf{P}^{14}$) Give A1 for 2 probabilities correct *(Max A1 if not at least 3dp)* *Tolerance* ± 0.0001
(iii)	$0.5684\times 0.8+0.2760$ $=0.731$	M1M1 A1 ft	**3**	For 0.5684×0.8 and 0.2760 *Accept 0.73 to 0.7312*
(iv)	$\begin{pmatrix}0.6 & 0.4 & 0 & 0\end{pmatrix}\mathbf{P}^{30}=(.\quad .\quad 0.4996\quad .)$ $\begin{pmatrix}0.6 & 0.4 & 0 & 0\end{pmatrix}\mathbf{P}^{31}=(.\quad .\quad 0.5103\quad .)$ Level 32	M1 A1 A1	**3**	Finding P(C) for some powers of **P** For identifying $\mathbf{P}^{31}$
(v)	Expected number of levels including the next change of location is $\frac{1}{0.2}=5$ Expected number of further levels in B is 4	M1 A1 A1	**3**	For $1/(1-0.8)$ or $0.8/(1-0.8)$ For 5 or 4 For 4 as final answer
(vi)	$\mathbf{Q}=\begin{pmatrix}0 & 0.7 & 0.1 & 0.2\\ 0.1 & 0.8 & 0 & 0.1\\ 0 & 0 & 0.9 & 0.1\\ 0.3 & 0.6 & 0.1 & 0\end{pmatrix}$ $\mathbf{Q}^n\to\begin{pmatrix}0.0916 & 0.6183 & 0.1908 & 0.0992\\ 0.0916 & 0.6183 & 0.1908 & 0.0992\\ 0.0916 & 0.6183 & 0.1908 & 0.0992\\ 0.0916 & 0.6183 & 0.1908 & 0.0992\end{pmatrix}$ A: 0.0916 B: 0.6183 C: 0.1908 D: 0.0992	B1 M1 M1 A2	**5**	*Can be implied* Evaluating powers of **Q** *or Obtaining (at least) 3 equations from* $\mathbf{pQ}=\mathbf{p}$ Limiting matrix with equal rows *or Solving to obtain one equilib prob* *or M2 for other complete method* Give A1 for two correct *(Max A1 if not at least 3dp)* *Tolerance* ± 0.0001

(vii)	$(0.11 \quad 0.75 \quad 0.04 \quad 0.1)\begin{pmatrix} 0 & 0.7 & 0.1 & 0.2 \\ 0.1 & 0.8 & 0 & 0.1 \\ a & b & c & d \\ 0.3 & 0.6 & 0.1 & 0 \end{pmatrix}$	M1	Transition matrix and $(0.11 \quad 0.75 \quad 0.04 \quad 0.1)$
	$=(0.11 \quad 0.75 \quad 0.04 \quad 0.1)$	A1	
	$0.075+0.04a+0.03=0.11$ $0.077+0.6+0.04b+0.06=0.75$ $0.011+0.04c+0.01=0.04$ $0.022+0.075+0.04d=0.1$	M1	Forming at least one equation *or* $a+b+c+d=1$
	$a=0.125,\quad b=0.325,\quad c=0.475,\quad d=0.075$	A2 **5**	Give A1 for two correct

4757 Further Applications of Advanced Mathematics (FP3)

General Comments

This paper was found to be much more accessible than those of previous years. Very many scripts contained substantially correct solutions to all three chosen questions, and about 40% of the candidates scored 60 marks or more (out of 72). The choice of topics was similar to last year, with questions 1 and 2 being considerably more popular than the others.

Comments on Individual Questions

1) *(Vectors)*
There were very many good answers to this question, and the average mark was about 18 (out of 24). Although some candidates used lengthy (yet valid) approaches, in general efficient methods were well known and applied competently to obtain the required results. The exception was part (iii), finding the shortest distance between two parallel lines, where a substantial number of candidates used the scalar product $\left|\mathbf{AB}\,.\,\hat{\mathbf{d}}\right|$ instead of $\left|\mathbf{AB}\times\hat{\mathbf{d}}\right|$.

2) *(Multi-variable calculus)*
This was the most popular question, attempted by about 85% of the candidates. It was also generally well answered, and the average mark was about 18. The partial derivatives were usually found correctly in part (i), with some candidates preferring to multiply out the expression for z first. Almost all knew how to find the stationary points in part (ii), and a good proportion obtained all three points correctly. In part (iii), errors were frequently made in the normal vector (usually with the sign of the z-component). Some candidates thought that the normal vector was the required answer to this part, and some gave the equation of the tangent plane instead of the normal line. The work on small changes in part (iv) was generally done well, although quite a number of candidates substituted the given coordinates into the equation of the surface and multiplied out, rarely making any worthwhile progress. Part (v) was sometimes omitted, but very many were able to find the required point and to show that there are no others. A fairly common error was to start with $\partial z/\partial x = -27$ instead of $\partial z/\partial x = 27$.

3) *(Differential geometry)*
This was the least popular question (attempted by about one third of the candidates), and it was also the worst answered question this year, with an average mark of about 14. A fair number of candidates seemed to be unfamiliar with the half-angle formulae, and so found several parts of the question to be inaccessible. In part (iii), most candidates could use their intrinsic equation to find the radius of curvature; those who used the parametric formula obtained a correct expression easily enough, but simplifying this to the given result proved to be challenging. Finding the centre of curvature in part (iv) was well understood and often done accurately; when marks were lost it was usually due to arithmetic slips or sign errors. Many candidates omitted part (v) (finding the curved surface area), and several others knew that they had to integrate $(1-\cos\theta)\cos\frac{1}{2}\theta$, but could not find a way of doing so.

4) *(Groups)*
The average mark on this question was about 16. Parts (ii), (v), (vi) and (vii) were answered very well, with candidates demonstrating a good understanding of groups. In part (i), many candidates lost marks for not showing enough working. Just asserting that the group *G* is generated by the element 3 is not sufficient; this should be established by listing all the powers of 3. Similarly in part (iv), the given result $\mathrm{ad}(x) = \mathrm{c}(x)$ was not always clearly shown to be true for all values of *x*. In part (iii), an explicit one-to-one correspondence between the elements of *G* and *H* was expected. Many candidates did not appear to understand the instruction 'specify an isomorphism' (which was quoted from the specification), and gave reasons why the two groups should be isomorphic.

5) *(Markov chains)*
This was the best answered question; the average mark was about 19, and about a third of the attempts scored full marks. Parts (i), (ii), (iv), (vi) and (vii) were all answered very well, with candidates demonstrating sound understanding of the techniques and confidence in using their calculators. In part (iii), the proportion who correctly used the diagonal elements in the transition matrix (to calculate the probability that the system remains in the same state) was higher than in previous years, but very many candidates found probabilities for level 14 and level 15 separately and assumed independence. Finding the run length in part (v) was sometimes omitted, but the correct formula $p / (1 - p)$ was quoted by a good proportion of the candidates. Some thought that they needed to add or subtract one from this value. In part (vi), several candidates preferred to form simultaneous equations and solve them to find the equilibrium probabilities, despite the suggestion in the question paper that they could simply consider a high power of the new transition matrix.